Feb. 21, 2014

Notation

$f'(x) = \frac{d}{dx} f(x)$ (function for Slope of tangent line at any pt)

Evaluated at a pt:

$f'(a) = \frac{d}{dx} f(x) \Big|_{x=a}$ means evaluated at $x = a$.

Multiple Derivatives

1st der: $f'(x) = \frac{d}{dx} f(x)$

2nd der: $f''(x) = \frac{d^2}{dx^2} f(x)$

3rd der: $f'''(x) = \frac{d^3}{dx^3} f(x)$

4th der: $f^{(4)}(x) = \frac{d^4}{dx^4} f(x)$

$\vdots$

Power Rule: $\frac{d}{dx} x^a = a x^{a-1}$

Ex: $\frac{d}{dx}\left(\frac{1}{x}\right)$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx} x^{-1} = -1 x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{h(x+h)x}$$

$$= \lim_{h \to 0} \frac{-h}{hx(x+h)}$$

$$= \lim_{h \to 0} -\frac{1}{x(x+h)}$$

$$= -\frac{1}{x^2}$$

Scalar Multiplication:

Ex: $\frac{d}{dx}(15x^2) = \lim_{h\to 0} \frac{15(x+h)^2 - 15x^2}{h}$

$= \lim_{h\to 0} 15\left(\frac{(x+h)^2 - 15x^2}{h}\right)$

$= 15 \underbrace{\lim_{h\to 0} \frac{(x+h)^2 - \cancel{15}x^2}{h}}_{=\frac{d}{dx}x^2} = 15\frac{d}{dx}x^2 = 30x$

THM: $f(x)$ diff'ble.

$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$

Proof: $\frac{d}{dx}cf(x) = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h\to 0} c\left(\frac{f(x+h)-f(x)}{h}\right)$

$= c \underbrace{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}_{\frac{d}{dx}f(x)}$ $\square$

Sums

$\frac{d}{dx}(x^2+x) = \lim_{h\to 0} \frac{(x+h)^2+(x+h)-x^2-x}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2+x+h-x^2-x}{h}$

$= \lim_{h\to 0} \frac{2xh+h^2+h}{h}$

$= \lim_{h\to 0} 2x+h+1$

$= 2x+1$

$= \frac{d}{dx}x^2 + \frac{d}{dx}x$

THM: $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

PF: $\frac{d}{dx}(f(x)+g(x)) = \lim_{h\to 0} \frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}$

$(b/c\ \frac{a+b}{c} = \frac{a}{c}+\frac{b}{c}) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$

$(\text{limit rule}) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)+g(x)}{h}$

$= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

Ex: $\frac{d}{dx}(x^{-3}-7x^2+6x^{-15}) = \frac{d}{dx}x^{-3} - 7\cdot\frac{d}{dx}x^2 + 6\frac{d}{dx}x^{-15}$

$= -3\cdot x^{-4} - 7\cdot 2x + 6(-15)x^{-16}$

Ex: $\frac{d}{dx}(\sqrt[3]{x} + 4\sqrt[13]{x}) = \frac{d}{dx}x^{1/3} + 4\frac{d}{dx}x^{1/13} = \frac{1}{3}x^{-2/3} + 4\left(\frac{1}{13}\right)x^{-12/13}$

Products? $\frac{d}{dx}f(x)g(x) = ?$

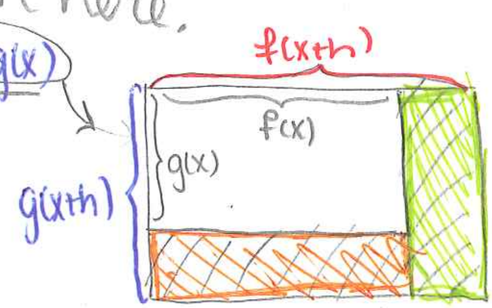
Not what you want it to.

Observe: $\frac{d}{dx}x^2 = \frac{d}{dx}x\cdot x$

$\frac{d}{dx}x = 1$. If $\frac{d}{dx}f(x)g(x) = \frac{d}{dx}f(x)\frac{d}{dx}g(x)$ then $\underbrace{\frac{d}{dx}x^2 = 1}_{\text{NOT TRUE}}$.

Something else is going on here.

$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$



shaded area is $f(x+h)g(x+h) - f(x)g(x)$

area of orange: $f(x)(g(x+h)-g(x))$

area of green: $g(x+h)(f(x+h)-f(x))$

$$\lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = \lim_{h\to 0} \frac{1}{h}\Big(f(x)\big(g(x+h)-g(x)\big) + g(x+h)\big(f(x+h)-f(x)\big)\Big)$$

$$= \lim_{h\to 0} f(x)\left(\frac{g(x+h)-g(x)}{h}\right) + \lim_{h\to 0} g(x+h)\left(\frac{f(x+h)-f(x)}{h}\right)$$

limits are nice w/ mult $\longrightarrow$ $= f(x)g'(x) + g(x)f'(x)$

THM: $\frac{d}{dx}\big(f(x)g(x)\big) = f(x)g'(x) + g(x)\cdot f'(x)$

Ex: $\frac{d}{dx}\big(\underbrace{(5x+2)}_{f(x)}\underbrace{(3x+1)}_{g(x)}\big) = \underbrace{(5x+2)}_{f(x)}\cdot \underbrace{3}_{g'} + \underbrace{(3x+1)}_{g}\cdot \underbrace{(5)}_{f'} = 15x+6+15x+5$

$= \boxed{30x+11}$

$f'(x)=5$

$g'(x)=3$

insert quotient rule here

Compositions: The chain Rule

Ex: $\frac{d}{dx}(5x+2)^{100}$

$f(x)=x^{100}$ $\quad f(g(x)) = (5x+2)^{100}$ $\quad f'(x)=100x^{99}$

$g(x)=5x+2$ $\quad g'(x)=5$

Ideally: $\frac{d}{dx}(5x+2) \neq 100\cdot 5^{99}$

NOT TRUE.

THM (Chain Rule):

If $f(x)$ & $g(x)$ are diff'ble then:

$$\frac{d}{dx}(f\circ g)(x) = \frac{d}{dx}\big(f(g(x))\big) = f'(g(x))\cdot g'(x)$$

Ex: $\frac{d}{dx}(5x+2)^{100} = \underbrace{100(5x+2)^{99}}_{f'(g(x))}\cdot\underbrace{5}_{g'(x)} = 500(5x+2)^{99}$

$f(x)=x^{100}$ $\quad f'(x)=100x^{99}$

$g(x)=5x+2$ $\quad g'(x)=5$

$f'(g(x)) = 100(g(x))^{99} = 100(5x+2)^{99}$

Why?

$$\frac{d}{dx} f(g(x)) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \cdot \frac{g(x+h)-g(x)}{g(x+h)-g(x)}$$

$$= \lim_{h\to 0} \underbrace{\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)}}_{\substack{\text{how } f \text{ changes vs.} \\ g(x) \text{ (instead of vs } x\text{)} \\ f'(g(x))}} \cdot \underbrace{\frac{g(x+h)-g(x)}{h}}_{g'(x)}$$

Leibniz: $\frac{d}{dx}(f(g(x))) = \frac{df}{dg}\cdot\frac{dg}{dx}$

Other Rules:

- Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

(it's really just the product rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\cdot\underbrace{\frac{1}{g(x)}}_{(g(x))^{-1}}\right) = f(x)(-1(g(x))^{-2}\cdot g'(x)) + f'(x)\cdot\frac{1}{g(x)}$$

$$= \frac{-f(x)g'(x)}{(g(x))^2} + \frac{f'(x)}{g(x)}$$

$$= \frac{-f(x)g'(x) + f'(x)g(x)}{(g(x))^2}$$

Ex: $$\frac{d}{dx}\frac{\overbrace{x-8}^{f(x)}}{\underbrace{x^2+1}_{g(x)}} = \frac{\overset{f'}{1}\cdot\overset{g}{(x^2+1)} - \overset{g'}{2x}\cdot\overset{f}{(x-8)}}{\underset{g^2}{(x^2+1)^2}}$$

Ex: $$\frac{d}{dx}\frac{\overbrace{(x^2+1)^2}^{f}}{\underbrace{(x+2)^{3/2}}_{g}} = \frac{(x^2+1)^2\cdot\frac{3}{2}(x+2)^{1/2} - 2(x^2+1)\cdot 2x\cdot(x+2)^{3/2}}{((x+2)^{3/2})^2}$$

Chain Rule

$f'(x) = 2(x^2+1)\cdot 2x \qquad g'(x) = \frac{3}{2}(x+2)^{1/2}\cdot 1$